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Multiple Solutions of the Periodic Boundary Value Problem for Some Forced Pendulum-Type Equations

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INTRODUCTION

The motivation of this paper is the study of the existence of multiple solutions for the problem

$$\begin{aligned} x'' + f(x)\,x' + a \sin x &= e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0. \end{aligned} \tag{0.1}$$

If $f(x) = c > 0$ and e is a constant $\bar{e}$, then integrating (0.1) over (0.2π) after multiplication of both members of the equation by x' shows that the only possible solutions are constant, and hence satisfy

$$a \sin x = \bar{e}.$$

Consequently, if we exclude the trivial translation by a multiple of 2π, (0.1) will have no solution if $|\bar{e}| > a$, one solution if $|\bar{e}| = a$ and two solutions if $|\bar{e}| < a$. In other terms, the intersection of the range of the operator $d^2/dt^2 + c(d/dt) \pm a \sin(\cdot)$ acting on 2π-periodic functions of class $\mathscr{C}^2$ with the subspace of constant functions in the space $C([0, 2\pi])$ of real continuous functions on $[0, 2\pi]$ is the closed interval $[-a, a]$, whose interior points are images of two distinct solutions and boundary points of one. We try to obtain similar results for the case where e is no more constant and f is much more general.

In Section 1, using upper and lower solutions techniques, we obtain results showing that, in the case of (0.1) with 2π-periodic f, if we write $e(t) = \bar{e} + \tilde{e}(t)$ with

$$\bar{e} = \frac{1}{2\pi} \int_0^{2\pi} e(t)\, dt, \qquad \int_0^{2\pi} \tilde{e}(t)\, dt = 0.$$

then, for each $\tilde{e}$, the set $\mathscr{R}(\tilde{e})$ of $\bar{e} \in \mathbb{R}$ for which (0.1) is solvable is a non-

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empty closed interval contained in $[-a, a]$. This extends previous results of Konecny [10], Castro [5] and Dancer [6]. In Section 2, we prove results implying that if $|f(x)| \geqslant c > 0$ for all $x \in \mathbb{R}$, then, for each $\tilde{e}$ and each

$$c > \sqrt{\frac{2}{3} \int_0^{2\pi} \tilde{e}^2(t)\, dt} = \sqrt{\frac{2}{3}}\, |\tilde{e}|_2$$

$$\mathscr{R}(\tilde{e}) \supset \left[-a \sin \frac{\pi}{2} \left(1 - \sqrt{\frac{2}{3}} \frac{|\tilde{e}|_2}{c}\right), a \sin \frac{\pi}{2} \left(1 - \sqrt{\frac{2}{3}} \frac{|\tilde{e}|_2}{c}\right)\right]$$

and for each $e = \bar{e} + \tilde{e}$ with $\bar{e}$ in this interval, the existence of at least two solutions is ensured. The proof depends on a degree argument. In Section 3, we refine and generalize an earlier result of Knobloch [9] and deduce the existence of at least two solutions when

$$|e|_\infty = \max_{t \in [0, 2\pi]} |e(t)| < a$$

and of one solution if equality holds. This implies that for $|\tilde{e}|_\infty \leqslant a$, $T(\tilde{e}) \supset [|\tilde{e}|_\infty - a, a - |\tilde{e}|_\infty]$. Degree theory is still the basic ingredient of this section.

In Section 4 we concentrate on the conservative case ($f \equiv 0$) and refine the results of [18] and [6] by showing through a mountain pass lemma argument the existence of at least two solutions for every e with mean value zero. Thus, in this case, $0 \in \mathscr{R}(\tilde{e})$ for every $\tilde{e}$ and it is shown in Section 5 that the set of $\tilde{e}$ for which $\mathscr{R}(\tilde{e})$ is a neighbourhood of 0 in $\mathbb{R}$ is open and dense in the subspace $\tilde{C}([0, 2\pi])$ of elements of $C([0, 2\pi])$ having mean value zero. Besides the uniform norm $|\cdot|_\infty$, we shall use the notation

$$|x|_p = \left(\int_0^{2\pi} |x(t)|^p\, dt\right)^{1/p}$$

for the L^p-norm and $|x|_{C^2}$ for the C^2-norm.

Let us mention that a short description of the role of the forced pendulum equation in the development of nonlinear analysis, and the corresponding references, can be found in [15], where special cases of some of the results given here can also be found.

1. Upper and Lower Solutions and the Range of Differential Equations of the Pendulum-Liénard Type

Let $F : [0, 2\pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $e : [0, 2\pi] \to \mathbb{R}$ be continuous. A lower (resp. upper) solution on $[0, 2\pi]$ of the differential equation

$$x''(t) = F(t, x(t), x'(t)) + e(t) \tag{1.1}$$

is a real C^2-function α (resp. β) on $[0, 2\pi]$ such that

$$\alpha''(t) \geqslant F(t, \alpha(t), \alpha'(t)) + e(t)$$
$$(\text{resp. } \beta''(t) \leqslant F(t, \beta(t), \beta'(t)) + e(t))$$

for all $t \in [0, 2\pi]$ (see, e.g., [8]).

The following lemmas will be useful.

LEMMA 1. *Assume that there exists some $T > 0$ such that*

$$F(t, x, y) = F(t, x + T, y) \tag{1.2}$$

for all $(t, x, y) \in [0, 2\pi] \times \mathbb{R} \times \mathbb{R}$. Then, if (1.1) *has a lower solution α_0 and an upper solution β_0, it also has a lower solution α and an upper solution β such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0, 2\pi]$.*

Proof. Take $\alpha = \alpha_0$ and $\beta = \beta_0 + kT$ with $k \in \mathbb{N}^*$ such that $k \geqslant T^{-1} \max_{t \in [0, 2\pi]} [\alpha_0(t) - \beta_0(t)]$.

Assume that F satisfies condition (1.2). We shall say that (1.1) satisfies a *Nagumo condition with respect to the boundary conditions*

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \tag{1.3}$$

if there exists $C > 0$ such that every solution of (1.1)–(1.3) satisfies the inequality

$$|x'|_\infty < C.$$

We refer to [8] and [14] for specific Nagumo conditions of more or less general nature, an important case being

$$|F(t, x, y) + e(t)| \leqslant \gamma(|y|)$$

on $[0, 2\pi] \times \mathbb{R} \times \mathbb{R}$, where γ is positive, continuous and such that

$$\int_0^\infty \frac{s\, ds}{\gamma(s)} = +\infty.$$

LEMMA 2. *Let $\bar{e}_1 \leqslant \bar{e}_2$ be real numbers, $\tilde{e} \in \tilde{C}([0, 2\pi])$ and let F satisfy* (1.2) *and be such that $F + \bar{e} + \tilde{e}$ satisfies a Nagumo condition with respect to the boundary conditions* (1.3) *for each $\bar{e}_1 \leqslant e \leqslant \bar{e}_2$. If the problems*

$$\begin{aligned} x'' &= F(t, x, x') + \bar{e}_i + \tilde{e}(t) \\ x(0) - x(2\pi) &= x'(0) - x'(2\pi) = 0 \qquad (i = 1, 2) \end{aligned} \tag{1.4}$$

have solutions then, for each $\bar{e}$ with

$$\bar{e}_1 \leqslant \bar{e} \leqslant \bar{e}_2, \tag{1.5}$$

the problem

$$\begin{aligned} x'' &= F(t, x, x') + \bar{e} + \tilde{e}(t) \\ x(0) - x(2\pi) &= x'(0) - x'(2\pi) = 0 \end{aligned} \tag{1.6}$$

has a solution.

Proof. Let $\bar{e}_1 \leqslant \bar{e} \leqslant \bar{e}_2$; if ξ_i is a solution of (1.4) ($i = 1, 2$), then, by (1.5), ξ_1 (resp. ξ_2) is an upper (resp. lower) solution of (1.6) satisfying the periodic boundary conditions. By Lemma 1, there will exist a lower solution α and an upper solution β of (1.6) with $\alpha(t) \leqslant \beta(t)$ and satisfying the boundary conditions (1.3). On the other hand, (1.6) clearly satisfies a Nagumo condition with respect to the boundary conditions (1.3) and α and β in the sense of [8] so that the existence of a solution x of (1.6) such that

$$\alpha(t) \leqslant x(t) \leqslant \beta(t)$$

for all $t \in [0, 2\pi]$ follows from classical results (see, e.g., [8]).

We now consider the problem

$$\begin{aligned} x'' + f(x)\, x' + g(t, x) &= e(t) \\ x(0) - x(2\pi) &= x'(0) - x'(2\pi) = 0, \end{aligned} \tag{1.7}$$

where $f: \mathbb{R} \to \mathbb{R}$, $g: [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ and $e: [0, 2\pi] \to \mathbb{R}$ are continuous and where, for some $T > 0$, one has

$$f(x + T) = f(x)$$

$$g(t, x + T) = g(t, x)$$

for all $t \in [0, 2\pi]$ and $x \in \mathbb{R}$. Notice that if we define F by

$$F(t, x, y) = -f(x)\, y - g(t, x),$$

then F satisfies (1.2) as well as a Nagumo condition with respect to the boundary conditions (1.3). Consequently, for each $\tilde{e} \in \tilde{C}([0, 2\pi])$, the set

$$\mathscr{R}(\tilde{e}) = \{\bar{e} \in \mathbb{R} : \text{the problem (1.7) with } e = \bar{e} + \tilde{e} \text{ has at least a solution}\}$$

is a (possible empty) interval of $\mathbb{R}$. We shall now obtain more information about $\mathbb{R}(\tilde{e})$.

LEMMA 3. *For each $\bar{x} \in \mathbb{R}$ and $\tilde{e} \in \tilde{C}([0, 2\pi])$, the problem*

$$\tilde{x}'' + f(\bar{x} + \tilde{x})\,\tilde{x}' + g(t, \bar{x} + \tilde{x}) - \frac{1}{2\pi}\int_0^{2\pi} g(s, (\bar{x} + \tilde{x}(s))\, ds = \tilde{e}(t),$$
$$\tilde{x}(0) - \tilde{x}(2\pi) = \tilde{x}'(0) = \tilde{x}(2\pi) = 0 \qquad (1.8)$$

has at least one solution in $\tilde{C}([0, 2\pi])$.

Proof. Define

$$H : \tilde{C}([0, 2\pi]) \to C^1([0, 2\pi]) \cap \tilde{C}([0, 2\pi]) = \tilde{C}^1([0, 2\pi])$$

by

$$(H\tilde{x})(t) = \int_0^t \tilde{x}(s)\, ds - \frac{1}{2\pi}\int_0^{2\pi}\int_0^t \tilde{x}(s)\, ds\, dt.$$

Notice that

$$\operatorname{Im} H \subset \{\tilde{x} \in \tilde{C}^1([0, 2\pi]) : \tilde{x}(0) - \tilde{x}(2\pi) = \tilde{x}'(0) - \tilde{x}'(2\pi) = 0\}.$$

Define $N_{\bar{x}} : \tilde{C}^1([0, 2\pi]) \to \tilde{C}^0([0, 2\pi])$ by

$$(N_{\bar{x}}\tilde{x})(t) = \tilde{e}(t) - f(\bar{x} + \tilde{x}(t))\,\tilde{x}'(t) - g(t, \bar{x} + \tilde{x}(t)) + \frac{1}{2\pi}\int_0^{2\pi} g(s, \bar{x} + \tilde{x}(s))\, ds.$$

Then (1.8) is equivalent to the fixed point problem

$$\tilde{x} = H^2 N_{\bar{x}}(\tilde{x})$$

in $\tilde{C}^1([0, 2\pi])$, and $H^2 N_{\bar{x}}$ is completely continuous in $\tilde{C}^1([0, 2\pi])$. Thus, by Leray–Schauder's theory [11] (see also [13]). (1.8) will have a solution if the set of possible solutions of

$$\tilde{x} = \lambda H^2 N_{\bar{x}}(\tilde{x}), \qquad \lambda \in [0, 1], \qquad (1.9)$$

is a priori bounded independently of λ. So let $\tilde{x}$ satisfy (1.9) for some $\lambda \in [0, 1]$. Then $\tilde{x} \in \tilde{C}^2([0, 2\pi])$ verifies the conditions (1.3) and the equation

$$\tilde{x}'' + \lambda f(\bar{x} + \tilde{x})\,\tilde{x}' + \lambda g(t, \bar{x} + \tilde{x}) - \frac{\lambda}{2\pi}\int_0^{2\pi} g(s, \bar{x} + \tilde{x}(s))\, ds = \lambda\tilde{e}(t). \qquad (1.10)$$

Integrating (1.10) over $[0, 2\pi]$ after multiplication by $\tilde{x}$, we get, after integration by parts and the use of (1.3),

$$\int_0^{2\pi} \tilde{x}'^2(t)\,dt = \lambda \int_0^{2\pi} [g(s, \bar{x} + \tilde{x}(s)) - \tilde{e}(s)]\,\tilde{x}(s)\,ds.$$

Hence, by well-known inequalities (see, e.g., [17, p. 208]) we get

$$|\tilde{x}|_\infty^2 \leqslant \frac{\pi^2}{6} |x'|_2^2 \leqslant \frac{\pi^2}{6} [2\pi\Gamma + |\tilde{e}|_1]\,|\tilde{x}|_\infty,$$

and hence

$$|\tilde{x}|_\infty \leqslant \frac{\pi^2}{6} [2\pi\Gamma + |\tilde{e}|_1] \tag{1.11}$$

and

$$|\tilde{x}'|_2 \leqslant \frac{\pi}{\sqrt{6}} [2\pi\Gamma + |\tilde{e}|_1] \tag{1.12}$$

with

$$\Gamma = \max_{\substack{t\in[0,2\pi]\\ x\in[0,T]}} |g(t, x)|.$$

Then integrating $|\tilde{x}''|$ taken from (1.10) and using (1.11) and (1.12) we obtain

$$|x''|_1 \leqslant C$$

where C depends only on $\tilde{e}$, Γ and $\max_{x\in[0,T]} |f(x)|$. As $\tilde{x}'$ necessarily vanishes at one point, this gives

$$|\tilde{x}'|_\infty \leqslant 2\pi C$$

and the possible solutions of (1.9) are a priori bounded in $\tilde{C}^1([0, 2\pi])$.

LEMMA 4. *For each $\tilde{e} \in \tilde{C}([0, 2])$, one has*

$$\mathscr{R}(\tilde{e}) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} g(t, \bar{x} + \tilde{x}_{\bar{x}}(t))\,dt : \tilde{x}_{\bar{x}} \text{ satisfies } (1.8) \text{ and } \bar{x} \in \mathbb{R} \right\} \neq \varnothing. \tag{1.13}$$

Proof. That the second set is non-empty follows from Lemma 3. If

$\bar{e} \in \mathscr{R}(\tilde{e})$, then (1.7) with $e = \bar{e} + \tilde{e}$ has at least one solution x and we can write $x = \bar{x} + \tilde{x}$ with $\tilde{x} \in \tilde{C}([0, 2\pi])$, Now (1.7) is equivalent to

$$\tilde{x}'' + f(\bar{x} + \tilde{x})\,\tilde{x}' + g(t, \bar{x} + \tilde{x}) - \frac{1}{2\pi}\int_0^{2\pi} g(s, \bar{x} + \tilde{x}(s))\,ds = \tilde{e}(t),$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(s, \bar{x} + \tilde{x}(s))\,ds = \bar{e}$$

so that $\bar{e}$ belongs to the set in the right-hand side of (1.13). Conversely, let $\bar{e}$ belong to this set; then there is an $\bar{x} \in \mathbb{R}$ and $\tilde{x}_{\bar{x}}$ verifying (1.8) and such that

$$\bar{e} = \frac{1}{2\pi}\int_0^{2\pi} g(t, \bar{x} + \tilde{x}_{\bar{x}}(t))\,dt.$$

Letting $x = \bar{x} + \tilde{x}_{\bar{x}}$, we have a solution of (1.7) with $e = \bar{e} + \tilde{e}$, so that $\bar{e} \in \mathscr{R}(\tilde{e})$ and the proof is complete.

We can now summarize and complete the above lemmas in the following

THEOREM 1. *For each $\tilde{e} \in \tilde{C}([0, 2\pi])$, $\mathscr{R}(\tilde{e})$ is a non-empty closed subinterval $[d(\tilde{e}), D(\tilde{e})]$ of $[\Gamma_-, \Gamma_+]$ where*

$$\Gamma_- = \frac{1}{2\pi}\int_0^{2\pi} \min_{x \in [0, T]} g(t, x)\,dt$$

$$\Gamma_+ = \frac{1}{2\pi}\int_0^{2\pi} \max_{x \in [0, T]} g(t, x)\,dt,$$

and, moreover

$$d(\tilde{e}) = \min_x \max_{t \in [0, 2\pi]} (x'' + f(x)\,x' + g(t, x) - \tilde{e}(t))$$

$$D(\tilde{e}) = \max_x \min_{t \in [0, 2\pi]} (x'' + f(x)\,x' + g(t, x) - \tilde{e}(t))$$

for $x \in C^2([0, 2\pi])$ and satisfying the boundary conditions (1.3).

Proof. We already know by Lemmas 2 and 4 that $\mathscr{R}(\tilde{e})$ is a non-empty subinterval of $[\Gamma_-, \Gamma_+]$. To prove that $\mathscr{R}(\tilde{e})$ is closed, let $(\bar{e}_k)$ be a sequence in $\mathscr{R}(\tilde{e})$ which converges to $\bar{e}$ and let x_k be a solution of (1.7) with $e = \bar{e}_k + \tilde{e}$ $(k \in \mathbb{N}^*)$. By the T-periodicity in x, we can assume without loss of generality that $\bar{x}_k \in [0, T]$. From (1.7), we see like in the proof of Lemma 3 that $\tilde{x}_k$ satisfies the fixed point equation

$$\tilde{x}_k = H^2 N_{\bar{x}_k}(\tilde{x}_k) \tag{1.14}$$

and that

$$|\tilde{x}_k|_\infty \leqslant \frac{\pi^2}{6}[2\pi\Gamma + |\tilde{e}|_1], \qquad |\tilde{x}_k'|_\infty \leqslant 2\pi C, \qquad |x_k''|_1 \leqslant C.$$

By the compactness of $[0, T]$ and the Ascoli–Arzela theorem there is a subsequence (x_{j_k}) such that $(\bar{x}_{j_k})$ converges to some $\bar{x} \in [0, 2\pi]$ and $(\tilde{x}_{j_k})$ converges in $\tilde{C}^1([0, 2\pi])$ to some $\tilde{x}$. From (1.14) we get

$$\tilde{x} = H^2 N_{\bar{x}}(\tilde{x})$$

and from

$$\frac{1}{2\pi}\int_0^{2\pi} g(t, \bar{x}_{j_k} + \tilde{x}_{j_k}(t))\,dt = \bar{e}_{j_k},$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} g(t, \bar{x} + \tilde{x}(t))\,dt = \bar{e}.$$

Consequently, $\bar{x} + \tilde{x}$ is a solution of (1.7) with $e = \bar{e} + \tilde{e}$ and $\mathscr{R}(\tilde{e})$ is closed.

Finally, as $\mathscr{R}(\tilde{e}) = [d(\tilde{e}), D(\tilde{e})]$, there exists $u \in C^2([0, 2\pi])$ and verifying (1.3) such that

$$u'' + f(u)\,u' + g(t, u) - \tilde{e}(t) = d(\tilde{e})$$

for all $t \in [0, 2\pi]$ and hence, for $x \in C^2([0, 2\pi])$ and verifying (1.3),

$$d(\tilde{e}) \geqslant d' = \inf_x \max_{t \in [0, 2\pi]} (x'' + f(x)\,x' + g(t, x) - \tilde{e}(t)).$$

If $d' < d(\tilde{e})$, then there will be d'' with $d' < d'' < d(\tilde{e})$ and there will be a $v \in C^2([0, 2\pi])$ and satisfying (1.3) such that

$$v'' + f(v)\,v' + g(t, v) - \tilde{e}(t) \leqslant d''.$$

Consequently, v is an upper solution for the problem

$$\begin{gathered} x'' + f(x)\,x' + g(t, x) = d'' + \tilde{e}(t), \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{gathered} \tag{1.15}$$

which satisfies (1.3). On the other hand,

$$u'' + f(u)\,u' + g(t, u) = d(\tilde{e}) + \tilde{e}(t) > d'' + \tilde{e}(t),$$

and hence u is a lower solution for (1.15) satisfying (1.3). By Lemma 1 and classical results (see, e.g., [8]), (1.15) will have a solution, a contradiction with the definition of $d(\tilde{e})$. The case of $D(\tilde{e})$ is treated similarly.

An immediate consequence of Theorem 1 is the following

COROLLARY 1. *For every continuous and 2π-periodic $f: \mathbb{R} \to \mathbb{R}$, every $a > 0$ and every $\tilde{e} \in \tilde{C}([0, 2\pi])$ the set $\mathscr{R}(\tilde{e})$ for the problem*

$$x'' + f(x)\,x' + a \sin x = \bar{e} + \tilde{e}(t)$$
$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

is a non-empty closed subinterval $[d(\tilde{e}), D(\tilde{e})]$ of $[-a, a]$ with

$$d(\tilde{e}) = \min_{x} \max_{t \in [0, 2\pi]} (x'' + f(x)\,x' + a \sin x - \tilde{e}(t))$$

$$D(\tilde{e}) = \max_{x} \min_{t \in [0, 2\pi]} (x'' + f(x)\,x' + a \sin x - \tilde{e}(t)).$$

for $x \in C^2([0, 2\pi])$ and verifying (1.3).

2. Norm Conditions on the Dissipative Term Ensuring the Existence of Distinct Periodic Solutions for Some Pendulum-Like Equation

We consider the problem

$$\begin{aligned} x'' + f(x)\,x' + g(x) &= e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0 \end{aligned} \tag{2.1}$$

where f and g are real continuous functions on $\mathbb{R}$ and $e \in L^2(0, 2\pi)$. If (2.1) has a solution, then, integrating over $(0, 2\pi)$ and using the boundary conditions, we obtain, if we write, for $y \in L^1(0, 2\pi)$,

$$\bar{y} = \frac{1}{2\pi} \int_0^{2\pi} y(t)\, dt, \qquad \tilde{y}(t) = y(t) - \bar{y}$$

$$\bar{e} = \frac{1}{2\pi} \int_0^{2\pi} g(x(t))\, dt,$$

and hence

$$\bar{e} \in \overline{\text{co}}\, g(\mathbb{R}),$$

where $\overline{\text{co}}$ denotes the convex closure.

We now prove the following

THEOREM 2. *Assume that the following conditions hold.*

(a) *There exists* $c > 0$ *such that*

$$|f(x)| \geqslant c \tag{2.2}$$

for every $x \in \mathbb{R}$.

(b) *There exist real numbers*

$$r < s \qquad \text{and} \qquad A \leqslant B$$

such that

$$\frac{1}{2\pi}\int_0^{2\pi} g(r+\tilde{y}(t))\,dt \leqslant A, \qquad \frac{1}{2\pi}\int_0^{2\pi} g(s+\tilde{y}(t))\,dt \geqslant B \tag{2.3}$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} g(r+\tilde{y}(t))\,dt \geqslant A, \qquad \frac{1}{2\pi}\int_0^{2\pi} g(s+\tilde{y}(t))\,dt \leqslant B$$

for every $\tilde{y} \in C^1([0, 2\pi])$ *having mean value zero, satisfying the boundary conditions in* (2.1) *and such that*

$$|\tilde{y}|_\infty \leqslant \frac{\pi}{\sqrt{6}}\frac{|\tilde{e}|_2}{c}.$$

Then (2.1) *has at least one solution if*

$$A \leqslant \bar{e} \leqslant B. \tag{2.4}$$

Proof. Let us consider, say, the case where (2.3) holds, the other one being similar. By classical arguments for which we refer to [13], it is sufficient to find an open bounded set Ω in $C^1([0, 2\pi])$ such that, for each $\lambda \in\,]0, 1[$, the possible solutions of the problem (with $\varepsilon \in\,]0, 1[$ fixed)

$$\begin{gathered} x'' + \lambda f(x)\,x' + (1-\lambda)\varepsilon\left(x - \frac{r+s}{2}\right) + \lambda g(x) = \lambda e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{gathered} \tag{2.5}$$

satisfy $x \notin \partial\Omega$, and such that $x'' + \varepsilon x = (\varepsilon/2)(r+s)$ has a 2π-periodic solution in Ω. To construct Ω, let $\lambda \in\,]0, 1[$ and let x be a possible solution of (2.5). Multiplying the equation by x', integrating over $(0, 2\pi)$ and using the boundary conditions, we obtain

$$\int_0^{2\pi} f(x(t))(x'(t))^2\,dt = \int_0^{2\pi} \tilde{e}(t)\,x'(t)\,dt.$$

As f is continuous and $|f|$ does not vanish, this implies, by (2.2) and Schwarz inequality,

$$c\,|x'|_2 \leqslant |\tilde{e}|_2$$

so that, by a well-known inequality,

$$|\tilde{x}|_\infty \leqslant \frac{\pi}{\sqrt{6}} \frac{|\tilde{e}|_2}{c}. \tag{2.6}$$

On the other hand, integrating (2.5) over $(0, 2\pi)$ we obtain

$$(1-\lambda)\,\varepsilon\left(\bar{x} - \frac{r+s}{2}\right) + \lambda\left[\frac{1}{2\pi}\int_0^{2\pi} g(\bar{x} + \tilde{x}(t))\,dt - \bar{e}\right] = 0. \tag{2.7}$$

But, using (2.6), (2.3) and (2.4), we get

$$\begin{aligned}&(1-\lambda)\,\varepsilon\left(r - \frac{r+s}{2}\right) + \lambda\left[\frac{1}{2\pi}\int_0^{2\pi} g(r + \tilde{x}(t))\,dt - \bar{e}\right]\\ &\quad\leqslant (1-\lambda)\,\varepsilon\left(\frac{r-s}{2}\right) + \lambda(A - \bar{e}) < 0\end{aligned} \tag{2.8}$$

$$\begin{aligned}&(1-\lambda)\,\varepsilon\left(s - \frac{r+s}{2}\right) + \lambda\left[\frac{1}{2\pi}\int_0^{2\pi} g(s + \tilde{x}(t))\,dt - \bar{e}\right]\\ &\quad\geqslant (1-\lambda)\,\varepsilon\left(\frac{s-r}{2}\right) + \lambda(B - \bar{e}) > 0.\end{aligned} \tag{2.9}$$

Now let

$$\Omega^0 = \left\{x \in C^1([0, 2\pi]) : r \leqslant \bar{x} \leqslant s \text{ and } |\tilde{x}|_\infty \leqslant \frac{\pi}{\sqrt{6}} \frac{|\tilde{e}|_2}{c}\right\}.$$

If x is a solution of (2.5) lying in Ω, then

$$|x|_\infty \leqslant \max(|r|, |s|) + \frac{\pi}{\sqrt{6}} \frac{|\tilde{e}|_2}{c} = C$$

and hence from the equality

$$\begin{aligned}&\int_0^{2\pi} (x''(t))^2\,dt + \lambda\int_0^{2\pi} f(x(t))\,x'(t)\,x''(t)\,dt\\ &\quad + (1-\lambda)\,\varepsilon\int_0^{2\pi}\left(x(t) - \frac{r+s}{2}\right)x''(t)\,dt + \lambda\int_0^{2\pi} g(x(t))\,x''(t)\,dt\\ &= \lambda\int_0^{2\pi} e(t)\,x''(t)\,dt,\end{aligned}$$

we deduce

$$|x''|_2 \leqslant C$$

where C depends only on c, r, s, $|\tilde{e}|_2$, f and g.
Consequently,

$$|x'|_\infty \leqslant \frac{\pi}{\sqrt{6}} D$$

for every possible solution of (2.5) lying in Ω^0. Define now Ω by

$$\Omega = \left\{ x \in C^1([0, 2\pi]) : t < \bar{x} < s, |\tilde{x}|_\infty < \frac{2\pi}{\sqrt{6}} \frac{|\tilde{e}|_2}{c}, |x'|_\infty < \frac{2\pi}{\sqrt{6}} C \right\}.$$

If $x \in \partial\Omega$, then necessarily $\bar{x} = r$ or s and if x satisfies (2.5), either (2.8) or (2.9) holds so that (2.7) cannot hold. Thus (2.5) has no solution on $\partial\Omega$ when $\lambda \in]0, 1[$. For $\lambda = 1$, if

$$\begin{gathered} x'' + \varepsilon x = \varepsilon(r + s)/2 \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0, \end{gathered} \tag{2.10}$$

then

$$\int_0^{2\pi} (x''(t))^2 \, dt - \varepsilon \int_0^{2\pi} (x'(t))^2 \, dt = 0$$

$$\bar{x} = \frac{r + s}{2}.$$

Consequently,

$$\varepsilon \int_0^{2\pi} (x'(t))^2 \, dt = \int_0^{2\pi} (x''(t))^2 \, dt \geqslant \int_0^{2\pi} (x'(t))^2 \, dt,$$

so that $x'(t) = 0$, and then

$$x = \bar{x} = \frac{r + s}{2} \in \Omega,$$

and the proof is complete.

COROLLARY 2. *Suppose that f satisfies the conditions of Theorem 1, $e \in L^2(0, 2\pi)$ and $a > 0$. Then, if*

$$c > \sqrt{\tfrac{2}{3}} \, |\tilde{e}|_2, \tag{2.11}$$

the problem

$$\begin{aligned} x'' + f(x)\,x' + a \sin x &= e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0 \end{aligned} \tag{2.12}$$

has at least one solution x *with* $\bar{x} \in [-\pi/2, \pi/2]$ *and one solution* x *with* $\bar{x} \in [\pi/2, 3\pi/2]$ *if*

$$|\bar{e}| \leqslant a \sin\left[\frac{\pi}{2}\left(1 - \sqrt{\frac{2}{3}}\,\frac{|\tilde{e}|_2}{c}\right)\right]. \tag{2.13}$$

Proof. We first take $r = -\pi/2$, $s = \pi/2$ in Theorem 1. Then, if $\tilde{y} \in C^1([0, 2\pi])$ is 2π-periodic, has mean value zero and is such that

$$|\tilde{y}|_\infty \leqslant \frac{\pi}{\sqrt{6}}\,\frac{|\tilde{e}|_2}{c},$$

we have, by (2.11),

$$|\tilde{y}|_\infty \leqslant \pi/2$$

and hence, using (2.13),

$$\sin\left(-\frac{\pi}{2} + \tilde{y}(t)\right) \leqslant \sin\left(-\frac{\pi}{2} + \frac{\pi}{\sqrt{6}}\,\frac{|\tilde{e}|_2}{c}\right) \leqslant -\frac{|\bar{e}|}{a} \leqslant \frac{\bar{e}}{a}$$

$$\sin\left(\frac{\pi}{2} + \tilde{y}(t)\right) \geqslant \sin\left(\frac{\pi}{2} - \frac{\pi}{\sqrt{6}}\,\frac{|e|_2}{c}\right) \geqslant \frac{|\bar{e}|}{a} \geqslant \frac{\bar{e}}{a}.$$

Consequently, the first part of the result follows from Theorem 2 with

$$A = a \sin\left(-\frac{\pi}{2} + \frac{\pi}{\sqrt{6}}\,\frac{|\tilde{e}|_2}{c}\right) = -B.$$

The second part is obtained by taking $r = \pi/2$, $s = 3\pi/2$ and noticing that

$$\begin{aligned} a \sin\left(\frac{3\pi}{2} + \tilde{y}(t)\right) &\leqslant a \sin\left(\frac{3\pi}{2} + \frac{\pi}{\sqrt{6}}\,\frac{|\tilde{e}|_2}{c}\right) \\ &= a \sin\left(-\frac{\pi}{2} + \frac{\pi}{\sqrt{6}}\,\frac{|\tilde{e}|_2}{c}\right) \leqslant -|\bar{e}| \leqslant \bar{e}. \end{aligned}$$

Remark 1. The above result shows, for the forced dissipative pendulum equation

$$x'' + cx' + a \sin x = e(t) = \bar{e} + \tilde{e}(t) \tag{2.14}$$

that if $\tilde{e}$ is given, then for each c with $|c| > \sqrt{\frac{2}{3}}\,|\tilde{e}|_2$, the set $\mathscr{R}(\tilde{e})$ of the $\bar{e}$ such that (2.14) has at least two 2π-periodic solutions contains the interval

$$\left[-a \sin\left[\frac{\pi}{2}\left(1-\sqrt{\frac{2}{3}}\frac{|\tilde{e}|_2}{c}\right)\right], a \sin\left[\frac{\pi}{2}\left(1-\sqrt{\frac{2}{3}}\frac{|\tilde{e}|_2}{c}\right)\right]\right].$$

3. Norm-Conditions on the Forcing Term Ensuring the Existence of Distinct Periodic Solutions for Some Pendulum-Like Equations

This section will be denoted to the study of the problem

$$\begin{aligned} x'' + h(t, x, x') + g(t, x) &= e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0 \end{aligned} \tag{3.1}$$

where h, g and e are real continuous functions on $[0, 2\pi] \times \mathbb{R}^2$, $[0, 2\pi] \times \mathbb{R}$ and $[0, 2\pi]$, respectively. We shall assume assume moreover that

$$h(t, x, 0) = 0 \tag{3.2}$$

for every $(t, x) \in [0, 2\pi] \times \mathbb{R}$ and that there exists some $T > 0$ such that

$$g(t, x + T) = g(t, x) \tag{3.3}$$

for every $(t, x) \in [0, 2\pi] \times \mathbb{R}$.

We shall say that h in problem (3.1) satisfies a *Nagumo-type condition* on $[r, s]$ if there exists a constant $C > 0$ such that for each $\lambda \in [0, 1]$ and each possible solution of

$$\begin{aligned} x'' + \lambda h(t, x, x') + \lambda g(t, x) &= \lambda e(t) \\ x(0) - x[2\pi] = x'(0) - x'(2\pi) &= 0 \end{aligned}$$

verifying $r \leqslant x(t) \leqslant s$ $(t \in [0, 2\pi])$, we have

$$|x'|_\infty < C.$$

Examples of admissible h are the following ones:

(a) h depends only on x' (see [12]);

(b) $|h(t, x, y)| \leqslant \gamma(|y|)$ for $(t, x, y) \in [0, 2\pi] \times [r, s] \times \mathbb{R}$ where γ is positive, continuous and such that

$$\int_0^\infty \frac{s\, ds}{\gamma(s)} = +\infty$$

(see, e.g., [8]).

We can now prove the following multiplicity result, which strengthens and generalizes an earlier existence result of Knobloch [9].

THEOREM 3. *Assume that, besides the above conditions on h and g, there exist real numbers r, s with $0 < s - r < T$, such that, for all $t \in [0, 2\pi]$, one has*

$$g(t, s) \leqslant e(t) \leqslant g(t, r) \tag{3.4}$$

and that h satisfies a Nagumo-type condition on $[r - T, s]$. Then (3.1) has a least one solution. If strict inequalities hold in (3.4), then (3.1) has at least two solutions not differing by a multiple of T.

Proof. By (3.2), (3.3) and (3.4), we have, for all integers k and l, and all $t \in [0, 2\pi]$,

$$e(t) - g(t, r + kT) - h(t, r + kT, 0) = e(t) - g(t, r) \leqslant 0,$$
$$e(t) - g(t, s + lT) - h(t, s + lT, 0) = e(t) - g(t, s) \geqslant 0,$$

with strict inequalities if they hold in (3.4). Taking first $k = l = 0$, the existence of at least one solution x of (3.1) satisfying

$$r \leqslant x(t) \leqslant s$$

for $t \in [0, 2\pi]$ follows from classical results (see, e.g., [8] and [12]). Now, if we define

$$\begin{aligned}
\operatorname{dom} L = \{x \in C^1([0, 2\pi]) : x \text{ is of class } C^2,\\
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\},\\
L : \operatorname{dom} L \subset C^1([0, 2\pi]) \to C^1([0, 2]), \qquad x \to x''\\
N : C^1([0, 2\pi]) \to C^0([0, 2\pi]), \qquad x \mapsto e - g(\cdot, x(\cdot)) - h(\cdot, x(\cdot), x'(\cdot)),\\
\Omega_{k,1} = \{x \in C^1([0, 2\pi]) : r + kT < x(t) < s + lT\\
\text{for } t \in [0, 2\pi] \text{ and } |x'|_\infty < C\}
\end{aligned}$$

$(k \leqslant 1)$, where C is the constant given by the Nagumo condition, a slight modification of the proof of Theorem V.5 in [8] shows that when the strict inequalities hold in (3.4), the following coincidence degrees exist and have the corresponding values, where d_B denotes the Brouwer degree (see, e.g., [13]) and

$$\Gamma : u \to \frac{1}{2\pi}\int_0^{2\pi} [e(t) - g(t, u)]\, du,$$

$$D_L(L-N, \Omega_{0,0}) = d_B(\Gamma,]r, s[, 0) = +1,$$

$$D_L(L-N, \Omega_{-1,-1}) = d_B(\Gamma,]r-T, s-T[, 0) = +1,$$

$$D_L(L-N, \Omega_{-1,0}) = d_B(\Gamma,]r-T, s[, 0) = +1.$$

But

$$\Omega_{0,0} \cap \Omega_{-1,-1} = \phi$$

and

$$\Omega_{0,0} \subset \Omega_{-1,0}, \qquad \Omega_{-1,-1} \subset \Omega_{-1,0}$$

so that the excision property of degree [13] implies

$$1 = D_L(L-N, \Omega_{-1,0}) = 2 + D_L(L-N, \Omega_{-1,0} \backslash (\bar{\Omega}_{-1,1} \cup \bar{\Omega}_{0,0}))$$

and hence

$$D_L(L-N, \Omega_{-1,0} \backslash (\bar{\Omega}_{-1,1} \cup \bar{\Omega}_{0,0})) = -1.$$

Consequently, (3.1) has a solution x in $\Omega_{-1,0} \backslash (\bar{\Omega}_{-1,1} \cup \bar{\Omega}_{0,0})$, i.e., a solution such that $r - T < x(t) < s$ for all $t \in [0, 2\pi]$, $x(r) > s - T$ for some $\tau \in [0, 2\pi]$, and $x(\tau') < r$ for some $\tau' \in [0, 2\pi]$. Consequently, this solution cannot differ from the one in $\Omega_{0,0}$ by a multiple of T, and the proof is complete.

COROLLARY 3. *Suppose that h satisfies the assuptions above and a Nagumo-type condition on $[-\pi/2, 3\pi/2]$. Then the problem (with $a > 0$)*

$$x'' + h(t, x, x') + a \sin x = e(t)$$

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

has at least two solutions not differing by a multiple of 2π if

$$|e|_\infty < a$$

and at least one solution if

$$|e|_\infty = a.$$

Proof. It suffices to take $g(t, x) = a \sin x$, $T = 2\pi$, $r = \pi/2$, $s = 3\pi/2$ in Theorem 3.

Remark 2. The above result shows, for the equation

$$x'' + cx' + a \sin x = e(t) = \bar{e} + \tilde{e}(t) \tag{3.5}$$

that if $\tilde{e}$ is given and if

$$|\tilde{e}|_\infty \leqslant a,$$

the set $\mathscr{R}(\tilde{e})$ of the $\bar{e}$ for which (3.5) has a 2π-periodic solution contains the interval $[|\tilde{e}|_\infty - a,\ a - |\tilde{e}|_\infty]$.

4. Multiple Solutions of Conservative Pendulum-Like Equation with Periodic Boundary Conditions and Forcing Term of Mean Value Zero

In this section we shall use variational methods to prove the existence of multiple solutions of the problem

$$\begin{aligned} &x'' + g(x) = \tilde{e}(t) \\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0, \end{aligned} \tag{4.1}$$

where $\tilde{e} \in \tilde{C}([0, 2\pi])$, $g : \mathbb{R} \to \mathbb{R}$ is continuous,

$$g(x + T) = g(x)$$

for all $x \in \mathbb{R}$ and some $T > 0$, and

$$\int_0^T g(x)\, dx = 0.$$

Consequently, the potential G of g defined by

$$G(x) = \int_0^x g(s)\, ds$$

is also T-periodic. If H is the (Sobolev) space of absolutely continuous real functions u on $[0, 2\pi]$ such that $u(0) = u(2\pi)$ and $u' \in L^2(0, 2\pi)$, with the inner product

$$(u, v)_H = \int_0^{2\pi} [u(t)\, v(t) + u'(t)\, v'(t)]\, dt,$$

define the C^1-functional $\varphi : H \to \mathbb{R}$ by

$$\varphi(u) = \int_0^{2\pi} [(1/2)\, u'^2(t) - G(u(t)) + \tilde{e}(t)\, u(t)]\, dt$$

so that

$$\varphi(u) = \int_0^{2\pi} [(1/2)\, \tilde{u}'^2(t) - G(\bar{u} + \tilde{u}(t)) + \tilde{e}(t)\, \tilde{u}(t)]\, dt.$$

It has been proved in [18] (and independently in [6]) that, under the above assumptions, (4.1) has a solution x which minimizes φ on H. We shortly recall the argument for completeness. For all $u \in H$, we have, if $\Gamma = \max_{x \in [0,T]} |G(x)|$,

$$\begin{aligned}\varphi(u) &\geqslant \tfrac{1}{2}\, |\tilde{u}'|_2^2 - 2\pi\Gamma - |\tilde{e}|_2\, |\tilde{u}|_2 \\ &\geqslant \tfrac{1}{2}\, |\tilde{u}'|_2^2 - |\tilde{e}|_2\, |\tilde{u}'|_2 - 2\pi\Gamma\end{aligned}$$

by the Wirtinger inequality so that $\varphi(u) \to +\infty$ if $|\tilde{u}|_{H^1} \to \infty$. If (u_k) is a minimizing sequence for φ, it follows from the relation

$$\varphi(u+T) = \varphi(u)$$

that we can assume without loss of generality that $\bar{u}_k \in [0, T]$. By the coercivity above, $(\tilde{u}_k)$ is bounded in H so that the same is true for (u_k). The result follows then from the weak lower semi-continuity of φ.

We shall show now that a multiplicity result holds in this situation. We denote by $D\varphi$ the gradient of φ and by $\langle\, ,\, \rangle$ the duality pairing between H and its dual H^*. We shall show that a multiplicity result holds in this situation. We denote by $D\varphi$ the gradient of φ.

THEOREM 4. *Under the above conditions, the problem* (4.1) *has at least two solutions which do not differ by a multiple of T.*

Proof. If the solution x which minimizes φ on H is not a strict local minimum, the result is proved. Thus we can assume now that x is a strict local minimum of φ and, to apply the Brézis–Coron–Nirenberg variant [4] of the Ambrosetti–Rabinowitz moutain pass lemma [2], we have to verify three conditions, which will ensure the existence of a critical value of φ different from $\varphi(x)$, and hence the result.

(i) *φ satisfies the weak Palais–Smale condition* (w–PS), i.e., if (u_k) is a sequence in H such that

$$\varphi(u_k) \to c \qquad \text{and} \qquad D\varphi(u_k) \to 0 \tag{4.2}$$

when $k \to \infty$, then c is a critical value of φ, i.e., there is a u such that $\varphi(u) = c$ and $D\varphi(u) = 0$.

Let $J : H \to H^*$ be the duality mapping. Then

$$\langle D\varphi(u), v \rangle = \langle Ju, v \rangle - \int_0^{2\pi} [u(t) + g(u(t)) - \tilde{e}(t)]\, v(t)\, dt$$

for every $u \in H$ and $v \in H$, and hence, if we define $N : H \to H^*$ by

$$\langle N(u), v \rangle = \int_0^{2\pi} [u(t) + g(u(t)) - \tilde{e}(t)]\, v(t)\, dt$$

for $u, v \in H$, we have $D\varphi = J - N$ and

$$N(u_k) \to N(u) \tag{4.3}$$

in H^* if $u_k \to u$ in H. Let now (u_k) be such that (4.2) holds. Without loss of generality, the T-periodicity of φ allows us to assume that $\bar{u}_k \in [0, T]$. On the other hand,

$$\begin{aligned} \varphi(u_k) &\geqslant \tfrac{1}{2} |\tilde{u}_k'|_2^2 - 2\pi\Gamma - |\tilde{e}|_2 |\tilde{u}_k|_2 \\ &\geqslant \tfrac{1}{2} |\tilde{u}_k'|_2^2 - |\tilde{e}|_2 |\tilde{u}_k'|_2 - 2\pi\Gamma \end{aligned}$$

so that, $(\varphi(u_k))$ being bounded, the same is true for $(|\tilde{u}_k'|_2)$ and hence for $|\tilde{u}_k|_H$, by Wirtinger's inequality. Going if necessary to a subsequence, we can thus assume that $u_k \rightharpoonup u$ in H. But then, by (4.3), $N(u_k) \to N(u)$ in H^* and, as $D\varphi(u_k) = J(u_k) - N(u_k) \to 0$, we see that

$$J(u_k) \to N(u)$$

in H^*. But then $u_k \to u$ in H and (4.2) implies

$$\varphi(u) = c, \qquad D\varphi(u) = 0,$$

i.e., c is a critical value of φ.

(ii) *There exists reals R, ρ, with $R > 0$, $\rho > \varphi(x)$ such that for all v with $|v - x| = R$, we have*

$$\varphi(v) \geqslant \rho.$$

As x is a strict local minimum, there exists $R > 0$ such that for all $v \neq x$ such that $|v - x|_H \leqslant R$, we have $\varphi(v) > \varphi(x)$. Notice that the T-periodicity implies that $R < 2\pi T$. Let

$$\rho = \inf_{|v - x|_H = R} \varphi(v) \geqslant \varphi(x) = \inf_H \varphi.$$

If $\rho = \varphi(x)$, there will exist a sequence (v_k) with $|v_k - x|_H = R$ and $|\varphi(v_k) - \varphi(x)| \leqslant 1/k^2$. For each k, we can apply to φ Ekeland's variational principle given in [7, Corollary 11] with $\varepsilon = 1/k^2$. We obtain in this way the existence of w_k such that

$$\begin{aligned} &\varphi(w_k) \leqslant \varphi(v_k) \leqslant \varphi(x) + 1/k^2 \\ &|w_k - v_k|_H \leqslant 1/k \\ &|D\varphi(w_k)|_{H^*} \leqslant 1/k \qquad (k \in \mathbb{N}^*). \end{aligned} \tag{4.4}$$

By the reasoning of part (i), there will exist a subsequence (w_{j_k}) of (w_k) and a $w \in H$ such that $w_{j_k} \to w$ if $k \to \infty$. Consequently, $v_{j_k} \to w$ and $|w - x|_H = R$. But, by (4.4), $\varphi(w) \leqslant \varphi(x)$, a contradiction. Thus $\rho > \varphi(x)$.

(iii) *There exists y such that $|y - x| > R$ and $\varphi(y) < \rho$.* If we take $y = x + T$, then $|y - x|_H = 2\pi T > R$ and $\varphi(y) = \varphi(x) < \rho$.

By the variant of the mountain pass lemma, there exists a critical value $c \geqslant \rho > \varphi(x)$. The corresponding critical point will be a weak, and hence classical, solution of (4.1) which cannot differ from x by a multiple of T.

COROLLARY 4. *For every $a > 0$ and $\tilde{e} \in \tilde{C}([0, 2\pi])$, the problem*

$$\begin{gathered} x'' + a \sin x = \tilde{e}(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{gathered}$$

has at least two solutions not differing by a multiple of 2π.

Proof. $g(x) = a \sin x$ satisfies all the cnditions of the above theorem with $T = 2\pi$.

Remark 3. The multiplicity result is optimal because the only 2π-periodic solutions of

$$x'' + \sin x = 0$$

are $k\pi$ $(k \in \mathbb{Z})$.

Remark 4. The condition

$$\int_0^{2\pi} \tilde{e}(t)\, dt = 0$$

is optimal without further restrictions on g because it is necessary and sufficient for the existence of a 2π-periodic solution for

$$x'' = \tilde{e}(t).$$

Remark 5. The idea of the proof can be used to obtain the following abstract result: if $\varphi : H \to \mathbb{R}$ is a C^1 functional such that

(i) φ satisfies the *Palais–Smale* condition [16], i.e., (PS) $(\varphi(u_k))$ bounded and $D\varphi(u_k) \to 0$ if $k \to \infty$ imply the existence of a convergent subsequence of (u_k),

(ii) c_0 is a local strict minimum,

(iii) $c_1 \leqslant c_0$ is a local minimum,

then φ has a critical value strictly greater than c_0.

This generalizes Amann [1] and Berger and Berger [3].

Remark 6. It is to be noticed that the functional φ in Theorem 4 does not satisy the Palais–Smale condition.

5. Properties of the Range of Conservative Pendulum-Like Operators with Periodic Boundary Conditions

Consider the problem

$$\begin{gathered} x'' + g(x) = \bar{e} + \tilde{e}(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{gathered} \tag{5.1}$$

where $\bar{e} \in \mathbb{R}$, $\tilde{e} \in \tilde{C}([0, 2\pi])$, $g : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic and has mean value zero. For each $\tilde{e} \in \tilde{C}([0, 2\pi])$, let $\mathscr{R}(\tilde{e})$ be the set of $\bar{e}$ such that (5.1) has a solution. By combining Theorems 1 and 4, we know that $\mathscr{R}(\tilde{e})$ is a closed interval, say,

$$\mathscr{R}(\tilde{e}) = [d(\tilde{e}), D(\tilde{e})]$$

and that

$$d(\tilde{e}) \leqslant 0 \leqslant D(\tilde{e}).$$

Let

$$\begin{aligned} \mathscr{S}_d &= \{\tilde{e} \in \tilde{C}([0, 2\pi]) : d(\tilde{e}) < 0\} \\ \mathscr{S}_D &= \{\tilde{e} \in \tilde{C}([0, 2\pi]) : D(e) > 0\}. \end{aligned}$$

Notice that for $g = 0$, $\mathscr{S}_d = \mathscr{S}_D = \phi$.

LEMMA 5. *$\mathscr{S}_d$ and $\mathscr{S}_D$ are open in $\tilde{C}([0, 2\pi])$.*

Proof. If they are empty the result is proved. If not, let $\tilde{e} \in \tilde{C}([0, 2\pi])$ be such that, say, $d(\tilde{e}) < 0$. By Theorem 1,

$$\min_u \max_{t \in [0,2\pi]} [u''(t) + g(u(t)) - \tilde{e}(t)] = d(\tilde{e}) < 0$$

and hence, if $\tilde{h} \in \tilde{C}([0, 2\pi])$ with $|\tilde{e} - \tilde{h}|_\infty < -d(\tilde{e})/2$, we have for some $u \in C^2([0, 2\pi])$ and verifying (1.3) such that

$$\begin{aligned} &u''(t) + g(u(t)) - \tilde{h}(t) \\ &\quad = u''(t) + g(u(t)) - \tilde{e}(t) + \tilde{h}(t) - \tilde{e}(t) \leqslant d(\tilde{e})/2 \leqslant 0 \end{aligned}$$

and thus $d(\tilde{h}) < 0$.

With a further restriction on g, we now have a density result for $\mathscr{S}_d$ and $\mathscr{S}_D$, and hence for their intersection

$$\mathscr{S} = \{\tilde{e} \in \tilde{C}([0, 2\pi]) : d(\tilde{e}) \cdot D(\tilde{e}) < 0\}.$$

THEOREM 5. *If g is of class C^1 and if its zeros are isolated, then $\mathscr{S}_d$, $\mathscr{S}_D$ and $\mathscr{S}$ are dense in $\tilde{C}([0, 2\pi])$.*

Proof. Consider, for definiteness, the case of $\mathscr{S}_d$. The zeros of g being isolated, we know that $0 \in \mathscr{S}_d$. Now, if $\mathscr{S}_d$ is not dense, there will exists $\tilde{h} \in \tilde{C}([0, 2\pi])$ and $r > 0$ such that, for all $\tilde{e} \in \tilde{C}([0, 2\pi])$ with $|\tilde{e} - \tilde{h}|_\infty < r$, we have $d(\tilde{e}) = 0$. By Theorem 4, there exists $x \in C^2([0, 2\pi])$ verifying (1.3) and

$$x'' + g(x) = \tilde{h}(t) \tag{5.2}$$

and hence there will be some $R > 0$ such that if $v \in C^2([0, 2\pi])$ with $|v - x|_{C^2} \leqslant R$, then

$$|v'' + g(v) - \tilde{h}|_\infty < r/2.$$

If for some $v \in C^2([0, 2\pi])$ verifying (1.3) and $|v - x|_{C^2} \leqslant R$, some $\tilde{e} \in \tilde{C}([0, 2\pi])$ and some $\varepsilon > 0$ we have

$$v'' + g(v) = -\varepsilon + \tilde{e}(t),$$

then

$$|-\varepsilon + \tilde{e} - \tilde{h}|_\infty = |v'' + g(v) - \tilde{h}|_\infty < r/2$$

and, as $\tilde{e} - \tilde{h}$ vanishes at one point $\tau \in [0, 2\pi]$,

$$\varepsilon = |-\varepsilon + \tilde{e}(\tau) - \tilde{h}(\tau)| \leqslant |-\varepsilon + \tilde{e} - \tilde{h}|_\infty < r/2.$$

Consequently,

$$|\tilde{e} - \tilde{h}|_\infty \leqslant |-\varepsilon + \tilde{e} - \tilde{h}|_\infty + \varepsilon < r,$$

so that we would have $d(\tilde{e}) < 0$ and $|\tilde{e} - \tilde{h}|_\infty < r$, a contradiction. Consequently, for all $v \in C^2([0, 2\pi])$, with $|v - x|_{C^2} \leqslant R$ and satisfying (1.3), we have

$$\int_0^{2\pi} [v''(t) + g(v(t))]\, dt \geqslant 0.$$

This implies, for all $h \in C^2([0, 2\pi])$ verifying (1.3),

$$\lim_{s \to 0} \int_0^{2\pi} s^{-1}[x''(t) + sh''(t) + g(x(t) + sh(t)) - x''(t) - g(x(t))]\, dt \geqslant 0$$

i.e.,

$$\int_0^{2\pi} g'(x(t))\, h(t)\, dt \geqslant 0.$$

Consequently,

$$g'(x(t)) = 0, \qquad t \in [0, 2\pi].$$

But then $g(x(\cdot))$ is constant, and hence, by taking the mean value of (5.2), $g(x(\cdot)) = 0$. Thus, x is constant, and, by (5.2), $\tilde{h} = 0$. But $0 \in \mathscr{S}_d$ so that $d(\tilde{h}) < 0$, a contradiction. The set $\mathscr{S}$, intersection of two open dense subsets, is therefore open and dense.

An immediate consequence of Theorem 5 is the following.

COROLLARY 5. *For problem*

$$x'' + a \sin x = \bar{e} + \tilde{e}(t)$$

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0,$$

$0 \in \mathscr{R}(\tilde{e}) = [d(\tilde{e}), D(\tilde{e})]$, *in which case there exist at least two solutions not differing from a multiple of* 2π *and the set*

$$\mathscr{S} = \{\tilde{e} \in \tilde{C}([0, 2\pi]) : d(\tilde{e}) \cdot D(\tilde{e}) < 0\}$$

is open and dense in $\tilde{C}([0, 2\pi])$.

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